

QUANTITATIVE ISOPERIMETRIC INEQUALITIES ON THE REAL LINE

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ABSTRACT. In a recent paper A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli have shown that, in the Gauss space, a set of given measure and almost minimal Gauss boundary measure is necessarily close to be a half-space.

Using only geometric tools, we extend their result to all symmetric log-concave measures μ on the real line. We give sharp quantitative isoperimetric inequalities and prove that among sets of given measure and given asymmetry (distance to half line, i.e. distance to sets of minimal perimeter), the intervals or complements of intervals have minimal perimeter.

Denote $d\gamma(t) = \exp(-t^2/2)dt/\sqrt{2\pi}$ the standard one-dimensional Gaussian measure. The classical Gaussian isoperimetric inequality [SC74] states that among sets of given measure in (\mathbb{R}^n, γ^n) half spaces have minimal Gauss boundary measure. This reads as $P_{\gamma^n}(\Omega) \geq J_\gamma(\gamma^n(\Omega))$, where J_γ is optimal (and defined later on in the text). In their paper [CFMP10] A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli have derived an improvement of the form

$$P_{\gamma^n}(\Omega) - J_\gamma(\gamma^n(\Omega)) \geq \Theta_{\gamma^n}(\gamma^n(\Omega), \lambda(\Omega)) \geq 0,$$

where $\lambda(\Omega)$ measures how far Ω is from a half-space. In their result the dependence in $\lambda(\Omega)$ is precise, whereas the dependence in $\gamma^n(\Omega)$ is not explicitly. In this paper we focus on the one dimensional case. Theorem 1.2 of A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli gives that

$$(1) \quad P_\gamma(\Omega) \geq J_\gamma(\gamma(\Omega)) + \frac{\lambda(\Omega)}{C(\gamma(\Omega))} \sqrt{\log(1/\lambda(\Omega))},$$

where $C(\gamma(\Omega))$ is a constant that depends only on $\gamma(\Omega)$. Our result (given in Theorem 2.6) is a sharp version of this statement, which is actually valid for **all symmetric log-concave measures** μ on the real line. This quantitative inequality implies that a set of given measure and almost minimal boundary measure is necessarily "close" to be half-line. This result stands not only for the Gaussian measure but for every measure satisfying a natural hypothesis \mathcal{H} (defined later), as proved in Theorem 3.5.

Organization of the paper. The outline of the paper is as follows: the first section recalls basic properties of the log-concave measures. The second part gives the main tool, named the *shifting lemma*, and establishes a *sharp quantitative isoperimetric inequality*. In the last section we provide (slightly weaker) estimates invoking *only classical functions*.

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1. THE ISOPERIMETRIC INEQUALITY ON THE REAL LINE

This section presents the *standard isoperimetric inequality* for the log-concave measures, and the *asymmetry* which measures the gap between a given set and the sets of minimal perimeter.

1.1. The standard isoperimetric inequality for the log-concave measures. Let μ be a measure with density function f . Throughout this paper, we assume that

- (i) the function f is supported and positive over some interval (a_f, b_f) , where a_f and b_f can be infinite,
- (ii) the measure μ is a *probability measure*,
- (iii) the measure μ is a *log-concave measure*,
- (iv) and the measure μ is symmetric with respect to the origin.

Observe that the point (iv) is not restrictive. As a matter of fact, the measure $\mu(\cdot + \alpha)$, where $\alpha \in \mathbb{R}$, shares the same isoperimetric properties as the measure μ . By the same token, the assumption (ii) is obviously not restrictive.

1.1.1. The μ -perimeter. Let Ω be a measurable set. Following [Fed69], define the set Ω^d of all points with *density* exactly $d \in [0, 1]$ as

$$\Omega^d = \left\{ x \in \mathbb{R}, \quad \lim_{\rho \rightarrow 0} \frac{\mathcal{L}^1(\Omega \cap B_\rho(x))}{\mathcal{L}^1(B_\rho(x))} = d \right\},$$

where \mathcal{L}^1 is the *Lebesgue measure* over the real line and $B_\rho(x)$ the ball with center x and radius ρ . Define the **essential boundary** $\partial^M \Omega$ as the set $\mathbb{R} \setminus (\Omega^0 \cup \Omega^1)$, consisting of points with neither empty nor full density. Define the **μ -perimeter** as

$$(2) \quad P_\mu(\Omega) = \mathcal{H}_\mu^0(\partial^M \Omega) = \int_{\partial^M \Omega} f(x) d\mathcal{H}^0(x),$$

where \mathcal{H}^0 is the *Hausdorff measure* of dimension 0 over \mathbb{R} and \mathcal{H}_μ^0 the measure of density f with respect to \mathcal{H}^0 . The **isoperimetric function** I_μ of the measure μ is defined by

$$(3) \quad I_\mu(r) = \inf_{\mu(\Omega)=r} P_\mu(\Omega).$$

In the log-concave case, we can give an explicit form to the isoperimetric function using the function J_μ .

1.1.2. The function J_μ . Denote F the cumulative distribution function of the measure μ . Since the function f is supported and positive over some interval (a_f, b_f) then the cumulative distribution function is increasing on the interval (a_f, b_f) . Define

$$(4) \quad J_\mu(r) = f(F^{-1}(r)),$$

where the quantile r ranges strictly from 0 to 1, $J_\mu(0) = J_\mu(1) = 0$, and F^{-1} denotes the inverse function of F .

1.1.3. The standard isoperimetric inequality. Following the article [Bob94] of S. G. Bobkov, since the measure μ is symmetric with respect to the origin, then the inverse function of F satisfies,

$$(5) \quad F^{-1}(r) = \int_{1/2}^r \frac{dt}{J_\mu(t)}, \quad \forall r \in (0, 1).$$

Using (5), one can check [Bob94] that the measure μ is log-concave **if and only if** J_μ is concave on $(0, 1)$. Furthermore it is known [Bor75] that the infima of (3) are exactly the intervals $(-\infty, \sigma_-)$ and $(\sigma_+, +\infty)$, where $\sigma_- = F^{-1}(r)$ and $\sigma_+ = F^{-1}(1 - r)$. The **isoperimetric inequality** states

$$(6) \quad P_\mu(\Omega) \geq J_\mu(\mu(\Omega)),$$

where Ω is a Lebesgue measurable set. This shows that, in the log-concave case, the isoperimetric function coincides with the function J_μ .

1.2. The asymmetry. We concern with quantifying the difference between any measurable set Ω and an isoperimetric infimum (i.e. any measurable set such that the isoperimetric inequality (6) is an equality) with the same μ -measure. Following [CFMP10], define the **asymmetry** $\lambda_\mu(\Omega)$ of a set Ω as

$$(7) \quad \lambda_\mu(\Omega) = \min \{ \mu(\Omega \Delta (-\infty, \sigma_-)), \mu(\Omega \Delta (\sigma_+, +\infty)) \},$$

where $\sigma_- = F^{-1}(\mu(\Omega))$ and $\sigma_+ = F^{-1}(1 - \mu(\Omega))$, and Δ is the symmetric difference operator.

Remark. The name asymmetry [FMP08] is inherited from the case of the Lebesgue measure on \mathbb{R}^n . In this case, the sets with minimal perimeter are balls, hence very symmetric.

Define the **isoperimetric projection** of a set Ω as the open half-line achieving the minimum in (7). In the case where this minimum is not unique we can chose whatever infima as an isoperimetric projection.

2. SHARP QUANTITATIVE ISOPERIMETRIC INEQUALITIES

This section gives a sharp improvement of (6) involving the asymmetry $\lambda(\Omega)$. In their paper [CFMP10] A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli use a technical lemma (Lemma 4.7, Continuity Lemma) to complete their proof. Their lemma applies in the n -dimensional case and is based on a compactness argument derived from powerful results in geometric measure theory. In the one-dimensional case, our approach is **purely geometric** and does not involve the continuity lemma.

2.1. The shifting lemma. The shifting lemma plays a key role in our proof. This lemma was introduced in [CFMP10] for the Gaussian measure. It naturally extends to even log-concave probability measures. For sake of readability, we begin with the shifting property.

Definition 1 (The shifting property) — *We say that a measure ν satisfies the **shifting property** when for every open interval (a, b) , the following is true:*

- *If $a + b \geq 0$ then for every (a', b') such that $a \leq a' < b' \leq +\infty$ and $\nu((a, b)) = \nu((a', b'))$, it holds $P_\nu((a, b)) \geq P_\nu((a', b'))$. In other words, if an interval is more to the right of 0, shifting it to the right with fixed measure, does not increase the perimeter.*
- *If $a + b \leq 0$ then for every (a', b') such that $-\infty \leq a' < b' \leq b$ and $\nu((a, b)) = \nu((a', b'))$, it holds $P_\nu((a, b)) \geq P_\nu((a', b'))$. In other words, if an interval is more to the left of 0, shifting it to the left with fixed measure, does not increase the perimeter.*

The following remark states that the shifting property can be equivalently formulated with the complement sets.

Remark. As the perimeter is complement-invariant, we may also shift "holes". The shifting property is equivalent to the following property.

- If $a + b \geq 0$ then for every (a', b') such that $a \leq a' < b' \leq +\infty$ and $v((a, b)) = v((a', b'))$, it holds $P_v((-\infty, a) \cup (b, +\infty)) \geq P_v((-\infty, a') \cup (b', +\infty))$.
- If $a + b \leq 0$ then for every (a', b') such that $-\infty \leq a' < b' \leq b$ and $v((a, b)) = v((a', b'))$, it holds $P_v((-\infty, a) \cup (b, +\infty)) \geq P_v((-\infty, a') \cup (b', +\infty))$.

Roughly, the next lemma shows that, for all measures such that the assumptions (i), (ii), and (iv) hold, the assumption (iii) is equivalent to the shifting property.

Lemma 2.1 (The shifting lemma) — *Every log-concave probability measure symmetric with respect to the origin has the shifting property.*

— *Conversely, let f be a continuous function, positive on an open interval and null outside. If the probability measure with density function f is symmetric with respect to the origin and enjoys the shifting property then it is log-concave.*

Proof. Let x, r be in $(0, 1)$ and t be in $(r/2, 1 - r/2)$. Define $\varphi(t) = J_\mu(t - r/2) + J_\mu(t + r/2)$. It represents the μ -perimeter of $(F^{-1}(t - r/2), F^{-1}(t + r/2))$ with measure equal to r . The function J_μ is symmetric with respect to $1/2$ since the density function f is supposed to be symmetric. As the function J_μ is concave and symmetric with respect to $1/2$, so is the function φ . In particular φ is non-decreasing on $(r/2, 1/2]$ and non-increasing on $[1/2, 1 - r/2)$. This gives the shifting property.

Conversely, let f be a continuous function, positive on an open interval and null outside. Define the isoperimetric function J_μ as in (4). We recall that μ is log-concave if and only if J_μ is concave on $(0, 1)$. Since the function J_μ is continuous, it is sufficient to have $J_\mu(x) \geq (1/2)(J_\mu(x - d) + J_\mu(x + d))$, for all $x \in (0, 1)$, where d is small enough to get $x - d \in (0, 1)$ and $x + d \in (0, 1)$. Let x and d be as in the previous equality. Since μ is symmetric, assume that $x \leq 1/2$. Put $a = F^{-1}(x)$, $b = F^{-1}(1 - x)$, $a' = F^{-1}(x + d)$, $b' = F^{-1}(1 - x + d)$, then (a', b') is a shift to the right of (a, b) . By the shifting property, we get $P_\mu((a, b)) \geq P_\mu((a', b'))$. The function J_μ is symmetric with respect to $1/2$, it yields (see Figure 1),

$$\begin{aligned} P_\mu((a, b)) &= J_\mu(x) + J_\mu(1 - x) = 2J_\mu(x), \\ P_\mu((a', b')) &= J_\mu(x + d) + J_\mu(1 - x + d) = J_\mu(x + d) + J_\mu(x - d). \end{aligned}$$

This ends the proof. \square

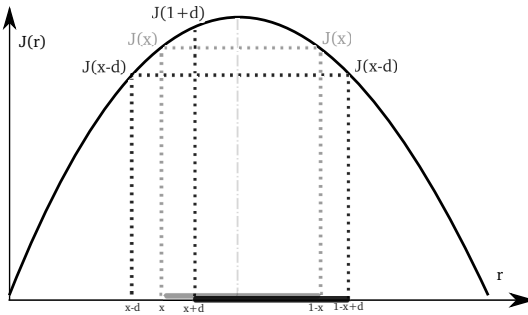


FIGURE 1. The log-concavity is equivalent to the shifting property

2.2. Lower bounds on the perimeter. In the following, we show that among sets of given measure and given asymmetry, the intervals or complements of intervals have minimal perimeter.

2.2.1. Structure of the sets with finite perimeter. Let Ω be a set of finite μ -perimeter. Consider $(K_k)_{k \in \mathbb{N}}$ a sequence of compact sets such that, for all $k \geq 0$, $K_0 \subset \dots \subset K_k \subset (-a_f, a_f)$ and $\cup_{k \in \mathbb{N}} K_k = (-a_f, a_f)$. Then, it yields

$$(8) \quad \Omega = \left(\bigcup_{k \in \mathbb{N}} (\Omega \cap K_k) \right) \cup E,$$

where E is such that $\mu(E) = 0$. Let k be an integer. On the compact K_k the function f is bounded from below by a positive real. Thus if $\Omega \cap K_k$ has finite μ -perimeter, so it has finite Lebesgue perimeter. As mentioned in [AFP00, Fed69], one knows that every set with finite Lebesgue perimeter can be written as at most countable union of open intervals and a set of measure equal to zero. It holds

$$\Omega \cap K_k = \left(\bigcup_{n \in I_k} (a_n, b_n) \right) \cup \mathcal{E}_k,$$

where I_k is at most countable, \mathcal{E}_k is such that $\mu(\mathcal{E}_k) = 0$, and (a_n, b_n) is such that

$$(9) \quad d\left((a_n, b_n), \bigcup_{l \in I_k \setminus \{n\}} (a_l, b_l)\right) > 0,$$

for all n in I_k and d the euclidean distance over the real line. Denote $\mathbb{1}_\Omega$ the indicator function of Ω and $\mathbb{1}'_\Omega$ its distribution derivative. The property (9) is a consequence of the fact that $\mathbb{1}'_\Omega$ is locally finite (see [Fed69] for instance). Since K_k is compact, the set I_k is finite. One can check that the decomposition (8) becomes

$$\Omega = \left(\bigcup_{n \in I} (a_n, b_n) \right) \cup \mathcal{E},$$

where I is at most countable, \mathcal{E} such that $\mu(\mathcal{E}) = 0$, and (a_n, b_n) such that

$$(10) \quad d\left((a_n, b_n), \bigcup_{k \in I \setminus \{n\}} (a_k, b_k)\right) > 0,$$

for all n in I . Notice that $\mu(\mathcal{E}) = 0$. Without loss of generality, assume that Ω is an *at most countable union of open intervals such that $\mathbb{1}'_\Omega$ is locally finite*.

2.2.2. Preliminaries. Let Ω be a set of finite μ -perimeter. As mentioned previously, assume that

$$\Omega = \bigcup_{n \in I} (a_n, b_n)$$

where I is an at most countable set and (10) holds. Suppose that

- an isoperimetric projection of Ω is $(-\infty, \sigma_-)$ (using a symmetry with respect to the origin if necessary),
- and that the measure of Ω is at most 1/2 (and we will see at the end of this section how to extend our result to larger measures).

Then the real number $\sigma_- = F^{-1}(\mu(\Omega))$ is non-positive. Denote $\sigma = -\sigma_-$. Since $\mathbb{1}'_\Omega$ is locally finite, there exists a finite number of sets (a_n, b_n) included in $(-\sigma, \sigma)$, it follows that

$$\Omega = \left(\bigcup_{h \in \Lambda_-} A_h \right) \cup I \cup \left(\bigcup_{h=1}^{N_-} A'_h \right) \cup \left(\bigcup_{h=1}^{N_+} B'_h \right) \cup J \cup \left(\bigcup_{h \in \Lambda_+} B_h \right),$$

where

- Λ_- and Λ_+ are at most countable sets;
- $A_h = (\alpha_{A_h}, \beta_{A_h})$ with $\beta_{A_h} \leq -\sigma$ (α_{A_h} can be infinite);
- I is either empty or of the form $I = (\alpha_I, \beta_I)$ with $\alpha_I \leq -\sigma < \beta_I$;
- A'_h is either empty or of the form $A'_h = (\alpha_{A'_h}, \beta_{A'_h})$ with $-\sigma < \alpha_{A'_h}$ and $\alpha_{A'_h} + \beta_{A'_h} < 0$;
- B'_h is either empty or of the form $B'_h = (\alpha_{B'_h}, \beta_{B'_h})$ with $\beta_{B'_h} < \sigma$ and $\alpha_{B'_h} + \beta_{B'_h} \geq 0$;
- J is either empty or of the form $J = (\alpha_J, \beta_J)$ with $\alpha_J < \sigma \leq \beta_J$;
- and B_h is either empty or of the form $B_h = (\alpha_{B_h}, \beta_{B_h})$ with $\alpha_{B_h} \geq \sigma$ (β_{B_h} can be infinite).

From Ω we build Ω_0 with same measure, same asymmetry, same isoperimetric projection, and lower or equal perimeter. Denote $L = \bigcup_{h \in \Lambda_-} A_h$ and $A_0 = (-\infty, \beta_{A_0})$ where $\beta_{A_0} = F^{-1}(\mu(L))$. Since $\mu(L) \leq \mu(\Omega)$, then $\beta_{A_0} \leq -\sigma$. Using the isoperimetric inequality (6) with L , it follows that $P_\mu(A_0) \leq P_\mu(L)$. The same reason gives that there exist a real number $\alpha_{B_0} \geq \sigma$ and a set $B_0 = (\alpha_{B_0}, +\infty)$ with lower or equal perimeter than $\bigcup_{h \in \Lambda_+} B_h$ (if non-empty). Shift to the left the intervals A'_h until they reach I or $-\sigma$. Shift to the right the intervals B'_h until they reach J or σ . The above operation did not change the amount of mass on left of $-\sigma$ and on the right of σ . We build a set Ω_0 with same asymmetry and same isoperimetric projection as Ω and lower or equal perimeter,

$$\Omega_0 = A_0 \cup I_0 \cup J_0 \cup B_0,$$

where

- $A_0 = (-\infty, \beta_0)$ with $\beta_{A_0} \leq -\sigma$;
- I_0 is either empty or of the form $I_0 = (\alpha_{I_0}, \beta_{I_0})$ with $\alpha_{I_0} \leq -\sigma < \beta_{I_0}$;
- J_0 is either empty or of the form $J_0 = (\alpha_{J_0}, \beta_{J_0})$ with $\alpha_{J_0} < \sigma \leq \beta_{J_0}$;
- and B_0 is either empty or of the form $B_0 = (\alpha_{B_0}, +\infty)$ with $\alpha_{B_0} > \sigma$.



FIGURE 2. The set Ω_0

2.2.3. Lowering the perimeter. A case analysis on the non-emptiness of sets I_0 and J_0 is required to obtain the claimed result. Every step described below lowers the perimeter (thanks to the shifting lemma, Lemma 2.1) and preserves the asymmetry. Before exposing this, we recall that the set Ω_0 is supposed to have $(-\infty, -\sigma)$ as an isoperimetric projection. Thus we pay attention to the fact that it is totally equivalent to ask either the asymmetry to be preserved or the quantity $\lambda(\Omega_0)/2 = \mu(\Omega_0 \cap (-\infty, -\sigma))$ to be preserved through all steps described below.

If I_0 and J_0 are both nonempty: Applying a symmetry with respect to the origin if necessary, assume that the center of mass of the hole between I_0 and J_0 is not less than 0. We can shift this hole to the right until it touches σ . Using the isoperimetric inequality (6), assume that there exist only one interval of the form $(\alpha'_{B_0}, +\infty)$ on the right of σ . We get the case where I_0 is nonempty and J_0 is empty.

If I_0 is nonempty and J_0 is empty: Shift the hole between A_0 and I_0 to the left until $-\infty$ (there exists one and only one hole between A_0 and I_0 since

Ω_0 is not a full measure set of $(-\infty, -\sigma)$). We shift the hole between I_0 and B_0 to the right until $+\infty$ (one readily checks that its center of mass is greater than 0). We get the only interval with same asymmetry and same isoperimetric projection as the set Ω_0 . This interval is of the form (the letter c stands for connected),

$$(11) \quad \Omega_c := (F^{-1}(\lambda(\Omega_0)/2), F^{-1}(\mu(\Omega_0) + \lambda(\Omega_0)/2)).$$

If J_0 is nonempty and I_0 is empty: Shift to the right the hole between J_0 and B_0 to $+\infty$ (there exists one hole between J_0 and B_0 since Ω_0 is not a full measure set of $(\sigma, +\infty)$). We obtain a set $A_0 \cup J'$ where J' is a neighborhood of σ .

- If $\mu(J') > \mu(A_0)$, then shift J' to the right (which has center of mass greater than 0) till $J' \cap (\sigma, +\infty)$ has weight equal to $\mu(A_0)$ (in order to preserve asymmetry). Using a reflection in respect to the origin, we find ourselves in the case where I_0 is nonempty and J_0 is empty.
- If $\mu(J') \leq \mu(A_0)$, then shift J' (which has center of mass greater than 0) to the right until $+\infty$ and get the case where I_0 and J_0 are both empty.

If I_0 and J_0 are both empty: Then the set Ω_0 is of the form (d stands for disconnected),

$$(12) \quad \Omega_d = (-\infty, F^{-1}(\mu(\Omega_0) - \lambda(\Omega_0)/2)) \cup (F^{-1}(1 - \lambda(\Omega_0)/2), +\infty).$$

Thus we proved the following lemma.

Lemma 2.2 — *Let Ω be a measurable set with μ -measure at most $1/2$ and $\lambda(\Omega)$ be the asymmetry of Ω . Then, it holds*

$$P_\mu(\Omega) \geq P_\mu(\Omega_c) \text{ or } P_\mu(\Omega) \geq P_\mu(\Omega_d),$$

where

- $\Omega_c = (F^{-1}(\frac{\lambda(\Omega)}{2}), F^{-1}(\mu(\Omega) + \frac{\lambda(\Omega)}{2}))$,
- $\Omega_d = (-\infty, F^{-1}(\mu(\Omega) - \frac{\lambda(\Omega)}{2})) \cup (F^{-1}(1 - \frac{\lambda(\Omega)}{2}), +\infty)$,

are sets with same measure and same asymmetry as Ω .

Let us emphasize that the sets Ω_c and Ω_d have fixed isoperimetric projection (i.e. $(-\infty, -\sigma)$), asymmetry, and measure. Observe that these properties are satisfied only for particular values of $\mu(\Omega)$ and $\lambda(\Omega)$.

2.2.4. Domains of sets with minimal perimeter given measure and given asymmetry.

We are concerned here with the domain $D = \{(\mu(\Pi), \lambda(\Pi)), \Pi \text{ measurable set}\}$.

The next lemma shows that asymmetry and perimeter are complement invariant.

Lemma 2.3 — *For every sets A and B with finite μ -perimeter the following is true:*

- the symmetric difference is complement-invariant: $A \Delta B = A^c \Delta B^c$,
- the asymmetry is complement-invariant: $\lambda(A) = \lambda(A^c)$,
- the perimeter is complement-invariant: $P_\mu(A) = P_\mu(A^c)$,
- and it holds $m(A) = m(A^c)$ where $m(A) = \min\{\mu(A), 1 - \mu(A)\}$.

Proof. One can check that the symmetric difference is complement-invariant (remark that $\mathbb{1}_{A \Delta B} = |\mathbb{1}_A - \mathbb{1}_B|$). The essential boundary is also complement-invariant, thus Definition 2 shows that the μ -perimeter is complement-invariant. Considering the symmetry of the isoperimetric function J_μ , we pretend that the isoperimetric projections are complements of the isoperimetric projections of the complement. This latter property and the fact that the symmetric difference is

complement-invariant give that the asymmetry is complement-invariant. The last equality is easy to check since μ is a probability measure. \square

Since asymmetry is complement-invariant, the domain D is symmetric with respect to the axis $x = 1/2$. Furthermore, we have the next lemma.

Lemma 2.4 — *For every measurable set Π , $0 \leq \lambda(\Pi) \leq \min(2m(\Pi), 1 - m(\Pi))$, where $m(\Pi) = \min\{\mu(\Pi), 1 - \mu(\Pi)\}$.*

Proof. Let Π be a measurable set. As asymmetry $\lambda(\Pi)$ and $m(\Pi)$ are complement-invariant (see Lemma 2.3), suppose that $\mu(\Pi) \leq 1/2$ thus $m(\Pi) = \mu(\Pi)$. Using symmetry with respect to the origin, suppose that $(-\infty, -\sigma)$ is an isoperimetric projection of Π (where $\sigma = -F^{-1}(\mu(\Pi))$).

We begin with the inequality $\lambda(\Pi) \leq 1 - \mu(\Pi)$. Since $(-\infty, -\sigma)$ is an isoperimetric projection of Π , it holds

$$\mu(\Pi \cap (\sigma, +\infty)) \leq \mu(\Pi \cap (-\infty, -\sigma)) = \mu(\Pi) - \lambda(\Pi)/2.$$

Remark that $\mu((-\sigma, \sigma)) = 1 - 2\mu(\Pi)$. Hence, $\lambda(\Pi)/2 = \mu(\Pi \cap (-\sigma, +\infty)) \leq 1 - 2\mu(\Pi) + \mu(\Pi) - \lambda(\Pi)/2$, which gives the expected result.

The inequality $\lambda(\Pi) \leq 2\mu(\Pi)$ can be deduced from

$$\lambda(\Pi)/2 = \mu((-\infty, -\sigma) \setminus \Pi) \text{ and } \mu((-\infty, -\sigma) \setminus \Pi) \leq \mu((-\infty, -\sigma)) = \mu(\Pi).$$

It is clear that $\lambda(\Pi) \geq 0$, this ends the proof. \square

By construction, the sets Ω_c and Ω_d verify three properties:

- (1) their measure is $\mu(\Omega)$,
- (2) their asymmetry is $\lambda(\Omega)$,
- (3) their isoperimetric projection is $(-\infty, -\sigma)$.

We recall that $\mu(\Omega) \leq 1/2$. Using Lemma 2.4, it is easy to check that Ω_c satisfies these properties if and only if

$$(13) \quad 0 \leq \lambda(\Omega) \leq \min(2\mu(\Omega), 1 - \mu(\Omega)).$$

Using the definition of the isoperimetric projection, one can check that Ω_d satisfies these properties **if and only if**

$$(14) \quad 0 \leq \lambda(\Omega) \leq \mu(\Omega).$$

Notice that on domain $0 \leq \lambda(\Omega) \leq \mu(\Omega)$ both sets exist. On this domain,

$$P_\mu(\Omega_d) - P_\mu(\Omega_c) = J_\mu(\mu(\Omega) - \lambda(\Omega)/2) - J_\mu(\mu(\Omega) + \lambda(\Omega)/2).$$

Since $\mu(\Omega) - \lambda(\Omega)/2 \leq \mu(\Omega) + \lambda(\Omega)/2 \leq 1 - \mu(\Omega) + \lambda(\Omega)/2$, we deduce from the concavity and the symmetry of the isoperimetric function that $P_\mu(\Omega_d) \leq P_\mu(\Omega_c)$ with equality **if and only if** $\mu(\Omega) = 1/2$. Using (13) and (14), we have the following result.

Lemma 2.5 — *Let Ω be a measurable set with μ -measure at most $1/2$ and $\lambda(\Omega)$ be the asymmetry of Ω . Then*

- the connected set of the form $\Omega_c = (F^{-1}(\lambda(\Omega)/2), F^{-1}(\mu(\Omega) + \lambda(\Omega)/2))$ has same measure and same asymmetry as Ω when $0 < \lambda(\Omega) \leq 1 - \mu(\Omega)$,
- and the disconnected set of the form

$$\Omega_d = (-\infty, F^{-1}(\mu(\Omega) - \lambda(\Omega)/2)) \cup (F^{-1}(1 - \lambda(\Omega)/2), +\infty)$$

has same asymmetry and same measure as Ω when $0 < \lambda(\Omega) \leq \mu(\Omega)$.

Besides, on the domain $0 < \lambda(\Omega) \leq \mu(\Omega)$, $P_\mu(\Omega_d) \leq P_\mu(\Omega_c)$ with equality **if and only if** $\mu(\Omega) = 1/2$.

2.2.5. *Isoperimetric deficit.* We end our case analysis with the following important result.

Theorem 2.6 — *Let Ω be a measurable set and $\lambda(\Omega)$ be the asymmetry of Ω . Set $m(\Omega) = \min\{\mu(\Omega), 1 - \mu(\Omega)\}$.*

- *If $0 < \lambda(\Omega) \leq m(\Omega)$ then*

$$(15) \quad P_\mu(\Omega) \geq J_\mu(m(\Omega) - \lambda(\Omega)/2) + J_\mu(\lambda(\Omega)/2),$$

- *If $m(\Omega) < \lambda(\Omega) \leq \min(2m(\Omega), 1 - m(\Omega))$ then*

$$(16) \quad P_\mu(\Omega) \geq J_\mu(m(\Omega) + \lambda(\Omega)/2) + J_\mu(\lambda(\Omega)/2),$$

and these inequalities are sharp.

Proof. Let Ω be a measurable set. If Ω has infinite μ -perimeter the result is true, hence assume that Ω has finite μ -perimeter. We distinguish four cases as illustrated in Figure 3.

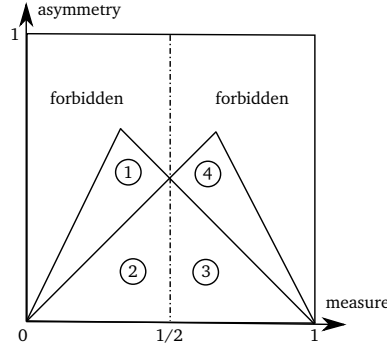


FIGURE 3. Domains of the sets with minimal perimeter given measure and asymmetry

If Ω has measure at most $1/2$, then $m(\Omega) = \mu(\Omega)$. Consider sets Ω_c defined in (11) and Ω_d defined in (12), compute

$$(17) \quad \begin{aligned} P_\mu(\Omega_d) &= J_\mu(\mu(\Omega) - \lambda(\Omega)/2) + J_\mu(\lambda(\Omega)/2), \\ P_\mu(\Omega_c) &= J_\mu(\mu(\Omega) + \lambda(\Omega)/2) + J_\mu(\lambda(\Omega)/2). \end{aligned}$$

Lemma 2.2 says that Ω has greater or equal μ -perimeter than Ω_c or Ω_d .

Domain 1: If $\mu(\Omega) < \lambda(\Omega) \leq 1 - \mu(\Omega)$ (and thus $m(\Omega) < \lambda(\Omega) \leq 1 - m(\Omega)$) then from Lemma 2.5 we know that Ω_d does not exist for such range of asymmetry. Necessary, it follows that $P_\mu(\Omega) \geq P_\mu(\Omega_c)$. Using (17), we complete (16).

Domain 2: If $0 < \lambda(\Omega) \leq \mu(\Omega)$ (and thus $0 < \lambda(\Omega) \leq m(\Omega)$) then from Lemma 2.5 we know that $P_\mu(\Omega_d) \leq P_\mu(\Omega_c)$. Thus $P_\mu(\Omega) \geq P_\mu(\Omega_d)$. Using (17), we get (15).

If Ω has measure greater than $1/2$, then $1 - \mu(\Omega) = m(\Omega)$. The Lemma 2.3 shows how to deal with sets of large measure and allows us to consider either Ω or its complement.

Domain 3: If $0 < \lambda(\Omega) \leq 1 - \mu(\Omega)$ (and thus $0 < \lambda(\Omega) \leq m(\Omega)$), the complement of Ω satisfies $0 < \lambda(\Omega^c) \leq \mu(\Omega^c)$ (Domain 2). Thus we know that $P_\mu(\Omega_d) \leq P_\mu(\Omega_c)$ (see the previous case on Domain 2). Finally, $P_\mu(\Omega) \geq P_\mu(\Omega_d^c)$ where Ω_d has same asymmetry and measure equal to $m(\Omega)$. Using (17), we complete (15).

Domain 4: If $1 - \mu(\Omega) < \lambda(\Omega) \leq \mu(\Omega)$ (and thus $m(\Omega) < \lambda(\Omega) \leq 1 - m(\Omega)$), the complement of Ω satisfies $\mu(\Omega^c) < \lambda(\Omega^c) \leq 1 - \mu(\Omega^c)$ (Domain 1). From the case on Domain 1, we know that $P_\mu(\Omega^c) \geq P_\mu(\Omega_c)$. Thus, $P_\mu(\Omega) \geq P_\mu(\Omega_c^c)$ where Ω_c has same asymmetry and measure equal to $m(\Omega)$. Using (17), we get (16).

This case analysis ends the proof. \square

The equalities (17) and the case analysis of the proof of Theorem 2.6 give the explicit lower bounds on μ -perimeter.

Proposition 2.7 — *The sets (see Figure 3)*

- $\Omega_c = (F^{-1}(\frac{\lambda}{2}), F^{-1}(\mu + \frac{\lambda}{2}))$, with $0 < \mu < \lambda \leq 1 - \mu$ and $\mu \leq 1/2$ (Domain 1),
- $\Omega_d = (-\infty, F^{-1}(\mu - \frac{\lambda}{2})) \cup (F^{-1}(1 - \frac{\lambda}{2}), +\infty)$, with $0 < \lambda \leq \mu$ and $\mu \leq 1/2$ (Domain 2),
- $\Omega_d^c = (F^{-1}(1 - \mu - \frac{\lambda}{2}), F^{-1}(1 - \frac{\lambda}{2}))$, with $0 < \lambda \leq 1 - \mu$ and $1/2 \leq \mu < 1$ (Domain 3),
- $\Omega_c^c = (-\infty, F^{-1}(\frac{\lambda}{2})) \cup (F^{-1}(1 - \mu + \frac{\lambda}{2}), +\infty)$, with $1 - \mu < \lambda \leq \mu$ and $1/2 \leq \mu < 1$ (Domain 4),

have the lowest perimeter given measure μ and given asymmetry λ .

Remark. The proof of Proposition 2.7 shows that, up to a negligible set, Ω_c , Ω_d , Ω_d^c and Ω_c^c are optimal given measure and given asymmetry. Moreover, it shows that the bounds in Theorem 2.6 are sharp.

3. SHARP ESTIMATE ON THE ASYMMETRY

In this section we use Theorem 2.6 to get a sharp estimate on the asymmetry. As a matter of fact, we show that a set of given measure and almost minimal boundary measure is necessarily close to be half-line.

3.1. The isoperimetric deficit function. We concern with an upper bound on the asymmetry of sets of given measure and given perimeter. Let Ω be a set with finite perimeter. Define the **isoperimetric deficit** of Ω as

$$(18) \quad \delta_\mu(\Omega) = P_\mu(\Omega) - J_\mu(\mu(\Omega)).$$

From the inequalities (15) and (16) of Theorem 2.6, we can compute a lower bound on the isoperimetric deficit as the asymmetry ranges from 0 to its upper bound $\min(2m(\Omega), 1 - m(\Omega))$ (see Lemma 2.4). Define the **isoperimetric deficit function** K_μ as follows.

- On $0 < y \leq x \leq 1/2$, set $K_\mu(x, y) = J_\mu(x - y/2) - J_\mu(x) + J_\mu(y/2)$.
- On $0 < x \leq 1/2$ and $x < y \leq \min(2x, 1 - x)$, set

$$K_\mu(x, y) = J_\mu(x + y/2) - J_\mu(x) + J_\mu(y/2).$$

The isoperimetric deficit function $K_\mu(x, y)$ is defined on the domain of all the possible values of $(m(\Omega), \lambda(\Omega))$. The isoperimetric deficit function gives the lower bounds found in Theorem 2.6. The next lemma focuses on the variations of K_μ .

Lemma 3.1 — *Let $0 < x \leq 1/2$. The function $y \mapsto K_\mu(x, y)$ is a non-decreasing lower semi-continuous function. Besides, it is concave on $x < y \leq \min(2x, 1 - x)$.*

Proof. The proof is essentially based on the concavity of J_μ .

On $0 < y \leq x$: Let $\Psi(t) = 1/2 (J_\mu(x/2 - t) + J_\mu(x/2 + t))$. Then the point $(x/2, \Psi(t))$ is the middle of the chord joining $(x/2 - t, J_\mu(x/2 - t))$ and $(x/2 + t, J_\mu(x/2 + t))$. Since J_μ is concave, it is well known that Ψ is a non-increasing function. Remark that $K_\mu(x, y) = 2\Psi(x/2 - y/2) - J_\mu(x)$, thus $y \mapsto K_\mu(x, y)$ is non-decreasing. Moreover the function is continuous as sum of continuous functions.

On $x < y \leq \min(2x, 1 - x)$: The function $y \mapsto K_\mu(x, y)$ is clearly concave as sum of two concave functions (thus continuous). On this domain,

$$(y/2) + (x + y/2) = x + y \leq x + \min(2x, 1 - x) \leq 1.$$

Hence the interval $\omega_y = (F^{-1}(y/2), F^{-1}(x + y/2))$ is on the left of the origin. Remark that $K_\mu(x, y) = P_\mu(\omega_y) - J_\mu(x)$. The shifting lemma (Lemma 2.1) applies here and shows that the function $y \mapsto K_\mu(x, y)$ is non-decreasing (as y increases, ω_y shifts to the right).

The variation at x is given by $K_\mu(x, x^+) - K_\mu(x, x) = J_\mu(3/2 x) - J_\mu(x/2)$, where $K_\mu(x, x^+) = \lim_{y \rightarrow x^+} K_\mu(x, y)$. One can check that $|1/2 - x/2| \geq |1/2 - 3x/2|$. Using the symmetry with respect to $1/2$ and the concavity of J_μ , one can check that $J_\mu(3/2 x) \geq J_\mu(x/2)$. Hence $K_\mu(x, x^+) \geq K_\mu(x, x)$.

This discussion shows that $y \mapsto K_\mu(x, y)$ is non-decreasing and lower semi-continuous on the whole domain. This ends the proof. \square

Defined the generalized inverse function of $y \mapsto K_\mu(x, y)$ as

$$K_{\mu, x}^{-1}(d) = \sup \{y \mid 0 \leq y \leq \min(2x, 1 - x) \text{ and } K_\mu(x, y) \leq d\}.$$

Lemma 3.1 shows that $y \mapsto K_\mu(x, y)$ is a non-decreasing lower semi-continuous function. It is easy to check that $K_{\mu, x}^{-1}$ is non-decreasing. Theorem 2.6 gives the next proposition.

Proposition 3.2 — *Let Ω be a measurable set and $\lambda(\Omega)$ be the asymmetry of Ω . Let $m(\Omega) = \min\{\mu(\Omega), 1 - \mu(\Omega)\}$ and $\delta_\mu(\Omega) = P_\mu(\Omega) - J_\mu(\mu(\Omega))$. It holds,*

$$(19) \quad \delta_\mu(\Omega) \geq K_\mu(m(\Omega), \lambda(\Omega)) \geq 0,$$

$$(20) \quad \text{and } \lambda(\Omega) \leq K_{\mu, m(\Omega)}^{-1}(\delta(\Omega)).$$

Proof. Since the asymmetry, the perimeter, the isoperimetric deficit, and $m(\Omega)$ are complement invariant, suppose that $m(\Omega) = \mu(\Omega) \leq 1/2$. Theorem 2.6 gives (19). Set $x = m(\Omega)$, the upper bound (20) is a consequence of the definition of $K_{\mu, x}^{-1}$ and (19). \square

3.1.1. The Gaussian case. We focus here on the Gaussian measure γ . Observe that $J_\gamma(t) \underset{t \rightarrow 0}{\sim} t \sqrt{2 \log(1/t)}$, so that $K_\gamma(x, y) \underset{y \rightarrow 0}{\sim} J_\gamma(\frac{y}{2}) \underset{y \rightarrow 0}{\sim} \frac{y}{2} \sqrt{2 \log(2/y)}$. In particular, there exists a constant $C(x)$ that depends only on x such that

$$K_\gamma(x, y) \geq \frac{y}{C(x)} \sqrt{\log(1/y)}, \quad \text{with } 0 \leq y \leq \min(2x, 1 - x),$$

we recover (1) from Proposition 3.2.

3.1.2. Lower bound on the isoperimetric deficit. In this section we give a convenient lower bound on the isoperimetric deficit. Define the function L_μ as follows.

- On $0 < y \leq x \leq 1/2$, set $L_\mu(x, y) = J_\mu(y/2) - y/(2x) J_\mu(x)$.
- On $0 < x \leq 1/2$ and $x < y \leq \min(2x, 1 - x)$, set

$$L_\mu(x, y) = J_\mu(y/2) - y/(2(1 - x)) J_\mu(x).$$

The next lemma shows that $0 \leq L_\mu \leq K_\mu$.

Lemma 3.3 — *Let Ω be a measurable set and $\lambda(\Omega)$ be the asymmetry of Ω . Let $m(\Omega) = \min \{\mu(\Omega), 1 - \mu(\Omega)\}$ and $\delta_\mu(\Omega) = P_\mu(\Omega) - J_\mu(\mu(\Omega))$. It holds,*

$$(21) \quad \delta_\mu(\Omega) \geq L_\mu(m(\Omega), \lambda(\Omega)) \geq 0,$$

Proof. Since the asymmetry, the perimeter, the isoperimetric deficit, and $m(\Omega)$ are complement invariant, suppose that $m(\Omega) = \mu(\Omega) \leq 1/2$. Set $x = m(\Omega)$ and $y = \lambda(\Omega)$.

On $0 < y \leq x$: Set $t = y/(2x - y)$ then $x - y/2 = t y/2 + (1 - t)x$. Since J_μ is concave, it holds

$$\begin{aligned} K_\mu(x, y) &= J_\mu\left(x - \frac{y}{2}\right) - J_\mu(x) + J_\mu\left(\frac{y}{2}\right) \geq (1 + t)J_\mu\left(\frac{y}{2}\right) - tJ_\mu(x), \\ &= \frac{1}{1 - y/2x} \left(J_\mu\left(\frac{y}{2}\right) - \frac{y}{2x} J_\mu(x)\right), \\ &\geq J_\mu\left(\frac{y}{2}\right) - \frac{y}{2x} J_\mu(x). \end{aligned}$$

As J_μ is concave, the function $t \mapsto J_\mu(t)/t$ is non-increasing and thus $(2/y)J_\mu(y/2) - (1/x)J_\mu(x) \geq 0$.

On $x < y \leq \min(2x, 1 - x)$: Using symmetry with respect to $1/2$, remark that

$$K_\mu(x, y) = J_\mu\left(x + \frac{y}{2}\right) - J_\mu(x) + J_\mu\left(\frac{y}{2}\right) = J_\mu\left((1 - x) - \frac{y}{2}\right) - J_\mu(1 - x) + J_\mu\left(\frac{y}{2}\right)$$

Substituting x with $1 - x$, the same calculus as above can be done.

This ends the proof. \square

The lower bound given in Lemma 3.3 is the key tool of the proof of the continuity theorem.

3.2. The continuity theorem. In the following, we improve the lower bound (20) on the asymmetry. We begin with a remark. Consider the *exponential* case where

$$J_{\exp}(t) = t \mathbb{1}_{[0, 1/2]} + (1 - t) \mathbb{1}_{[1/2, 1]}.$$

One gets $K_{\exp} = 0$ on $0 \leq y \leq x \leq 1/2$. This means that there exists sets with a positive asymmetry and an isoperimetric deficit null. In the case of the *exponential-like distributions* (defined later on), the intervals $(-\infty, F^{-1}(r))$ and $(F^{-1}(1 - r), +\infty)$ are not the only sets with minimal perimeter (up to a set of measure equals to 0) given measure r .

We specify this thought defining a natural hypothesis (\mathcal{H}) . Furthermore, we prove that the asymmetry goes to zero as the isoperimetric deficit goes to zero.

3.2.1. The hypothesis \mathcal{H} . We can get a better estimate on the asymmetry making another hypothesis. From now, suppose that the measure μ is such that

$$(\mathcal{H}) \quad \exists \varepsilon > 0 \quad \text{s.t.} \quad t \mapsto J_\mu(t)/t \text{ is decreasing on } (0, \varepsilon).$$

This hypothesis means that J_μ is non-linear in a neighborhood of the origin. We can be more specific introducing the property:

$$(\overline{\mathcal{H}}) \quad \exists \varepsilon > 0 \text{ and } c > 0 \quad \text{s.t.} \quad J_\mu(t) = c t, \quad \forall t \in [0, \varepsilon].$$

Since $t \mapsto J_\mu(t)/t$ is non-increasing, it is not difficult to check that $(\overline{\mathcal{H}})$ is the alternative hypothesis of (\mathcal{H}) .

3.2.2. *The exponential-like case.* The exponential tail measures can be defined by the following property:

$$(\mathcal{E}xp) \quad \exists \tau > 0 \text{ and } c, c' > 0 \text{ s.t. } f(t) = c' \exp(ct), \forall t \in (-\infty, \tau).$$

Proposition 3.4 — *The property $(\overline{\mathcal{H}})$ is equivalent to the property $(\mathcal{E}xp)$.*

Proof. The proof is essentially derived [Bob94] from the equality $(F^{-1})'(t) = 1/J_\mu(t)$, for all $t \in (0, 1)$. Suppose that the measure satisfies $(\overline{\mathcal{H}})$. Using the above equality for sufficiently small values of r , one can check that $F^{-1}(r) = \frac{1}{c} \log(r) + c''$, where c'' is a constant. Hence $F(x) = \exp(c(x - c'')) = \frac{c'}{c} \exp(cx)$, which gives the property $(\mathcal{E}xp)$. Conversely, suppose that the measure satisfies $(\mathcal{E}xp)$. A simple computation gives the property $(\overline{\mathcal{H}})$. \square

Suppose that μ satisfies $(\overline{\mathcal{H}})$. It is not difficult to check that the sets (and their symmetric) $(-\infty, F^{-1}(r - s)) \cup (F^{-1}(s), +\infty)$, for all $s \in (0, \min(\epsilon, r))$, have minimal perimeter given measure r . It would be natural to define the asymmetry with these sets (and not only $(-\infty, -\sigma)$ and $(\sigma, +\infty)$). In this case, our asymmetry (defined by (7)) is not relevant in terms of continuity.

3.2.3. *Continuity of the asymmetry for non-exponential distributions.* The hypothesis (\mathcal{H}) ensures that the distribution is non-exponential. It is the right framework dealing with continuity as shown in the next theorem.

Theorem 3.5 (Continuity) — *Assume that the measure μ satisfies the assumption \mathcal{H} , then the asymmetry goes to zero as the isoperimetric deficit goes to zero.*

Proof. The proof is based on Lemma 3.3 and Proposition 3.2. Let $u, v \in (0, 1)$, define $\rho(u, v) = J_\mu(u)/u - J_\mu(v)/v$. Suppose $u < v$. Since J_μ is concave, it is easy to check that if $\rho(u, v) = 0$, then $\forall u' \leq u$, $\rho(u', v) = 0$. In particular \mathcal{H} implies that $\forall u < v$, $\rho(u, v) > 0$, for sufficiently small values of v . Remark that $L_\mu(x, y) = (y/2)\rho(y/2, x)$ if $0 < y \leq x$, and $L_\mu(x, y) = (y/2)\rho(y/2, 1 - x)$ if $x < y \leq \min(2x, 1 - x)$. Hence \mathcal{H} implies that $L_\mu > 0$. Using Lemma 3.3, it yields that $K_\mu > 0$.

Finally, it is easy to check that if $K_\mu > 0$ then there exists a neighborhood of 0 such that K_μ is increasing. Taking a sufficiently small neighborhood if necessary, one can suppose that K_μ is continuous (the only point of discontinuity of K_μ is $y = x$). On this neighborhood, $K_{\mu, x}^{-1}$ is a continuous increasing function. Using (20), this gives the expected result. \square

Roughly, it uncovers that a set of given measure and almost minimal boundary measure is necessarily close to be a half-line. Moreover we recover the following well-known result.

Corollary — *Assume that the measure μ satisfies the assumption \mathcal{H} , then the half-lines are the only sets of given measure and minimal perimeter (up to a set of μ -measure null).*

This last results ensure that the asymmetry is the relevant notion speaking of the isoperimetric deficit.

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